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Terminal Model of Newtonian Dynamics

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A new type of dissipation function which does not satisfy the Lipschitz condition at equilibrium states is proposed. Newtonian dynamics supplemented by this dissipation function becomes irreversible and has a well-organized probabilistic structure.

1. INTRODUCTION

Classical dynamics describes processes in which the direction of time does not matter: its governing equations are invariant with respect to time inversion, in the sense that the time-backward motion can be obtained from the governing equations by time inversion, $t \to -t$. As stressed by Prigogine (1980), in this view future and past play the same role: nothing can appear in the future which could not already exist in the past since the trajectories followed by particles can never cross. This means that classical dynamics cannot explain the emergence of new dynamical patterns in nature in the same way in which nonequilibrium thermodynamics does. That is why the discovery of chaotic motions (which could lead to unpredictability in classical dynamics) has shaken up the scientific community, and the number of publications in the area of chaos is still growing. However, is this a key to the problem of unpredictability and irreversibility in Newtonian dynamics? In my opinion, the answer is no, since "chaotic" dynamical equations do not "generate" randomness: they are rather driven by random initial conditions.

Indeed, let us consider a steady laminar flow whose instability is characterized by an exponential multiplier:

$$\tilde{v} = v_0 \, e^{\mu t}, \qquad 0 < \mu < \infty \tag{1}$$

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Obviously, the solution with infinitely close initial condition

$$\tilde{v} = v_0 + \varepsilon, \qquad \varepsilon \to 0 \tag{2}$$

will remain infinitely close to the original one:

$$|\tilde{v} - v_0| = \varepsilon\, e^{\mu t} \to 0 \qquad \text{if} \quad \varepsilon \to 0, \quad t \le N < \infty \tag{3}$$

during all the bounded time intervals. This means that random solutions can result only from random initial conditions when ε in equation (2) is small, but finite rather than infinitesimal.

The same arguments can be applied to discrete chaotic systems if the divergence of actual trajectories in (1) is replaced by the divergence of trajectories in configuration space.

Thus, as in stochastic differential equations, the changes in initial conditions for chaotic equations must be finite, although they may be humanly indistinguishable. However, unlike stochastic equations, the phenomenon of unpredictability in chaotic systems has a different origin: it is caused by exponential amplifications of the initial changes due to the mechanism of instability. Indeed, if two trajectories initially are "very close" and then they diverge exponentially, the same initial conditions can be applied to either of them, and therefore the motion cannot be traced.

But then two arguments can be brought up. First, from the mechanical viewpoint, stability is not an invariant of motion: it depends upon the frame of reference. For instance, the same inviscid flow can be stable in the Eulerian representation and unstable in the Lagrangian one (Arnold, 1988) or in a frame of reference moving with the streamlines (Zak, 1990*c*). This leads to the following question: is it possible to find such a (noninertial) frame of reference in which the inertia forces would stabilize the motion, i.e., eliminate all the positive Lyapunov exponents? The answer to that question was given in Zak (1985*a*). I introduced a specially selected rapidly oscillating frame of reference in which the originally chaotic motion was stabilized by inertia forces coupled with the motion itself. In other words, I found a frame of reference which provides the best "view" of the motion. However, there was a certain price paid for this representation: the component of the solution corresponding to the transport motion with the frame of reference contained the function $\sin \omega t$, $\omega \to \infty$, which is actually multivalued. Indeed, for any arbitrarily small interval Δt there always exists such a large frequency $\omega > \Delta/2\pi$ that within this interval the function runs through all its values. This means that in order to eliminate chaos one has to enlarge the class of smooth functions by introducing nondifferentiable functions, and that leads us to the second question: is chaos an invariant of motion or is it an attribute of a mathematical model? From the mathematical viewpoint the concept of stability is related to a certain class of function, or a type of space, and therefore, the same solution can be stable in one space and unstable in another, depending upon the "distance" between two solutions. Hence, the occurrence of chaos in the description of mechanical motions means only that these motions cannot be properly described by smooth functions if the scale of observations is limited. These arguments can be linked to Godel's (1931) incompleteness theorem and the Richardson (1968) proof that the theory of elementary functions in classical analysis is undecidable. Indeed, classical dynamics, in addition to Newton's laws, is based upon certain assumptions of a purely mathematical nature. They restrict the class of functions that describe the motions to functions of sufficient smoothness. Such an artificial limitation, which does not follow from the axioms of mechanics, may become inconsistent with the physical nature of the motions. As shown in Zak (1974, 1982*a–c*, 1985*a,b*), these inconsistencies lead to instabilities (in the class of smooth functions) of the equations which govern turbulent and chaotic motions.

The first step toward enlarging of the class of functions for modeling turbulence was made by Reynolds (1895), who decomposed the velocity field into mean and pulsating components, and actually introduced a multivalued velocity field. However, this decomposition brought new unknowns without additional governing equations, and that created a closure problem. In Zak (1986*a,b*) it was shown that the Reynolds equations can be obtained by referring the Navier–Stokes equations to a rapidly oscillating frame of reference, while the Reynolds stresses represent the contribution of inertia forces. From these viewpoint the closure has the same status as the proof of Euclid's parallel postulate, since the motion of the frame of reference can be chosen arbitrarily. In other words, the closure of the Reynolds equations represents a case of undecidability in classical mechanics. However, based upon the interpretation of the Reynolds stresses as inertia forces, it is reasonable to choose the motion of the frame of reference such that the inertia forces eliminate the original instability. In other words, the enlarged class of functions should be selected such that the solution to the original problem in that class of functions will not possess an exponential sensitivity to changes in initial conditions. This stabilization principle has been formulated and applied to chaotic and turbulent motions (Zak, 1984, 1985, 1986*a,b*, 1989*a*). As shown there, the motions which are chaotic (or turbulent) in the original frame of reference can be represented as a sum of the mean motion and rapid fluctuations, while both components are uniquely defined. It is worth emphasizing that the amplitude of velocity fluctuation is proportional to the degree of the original instability, and therefore the rapid fluctuations can be associated with the measure of the uncertainty in the description of the motion. It should be noticed that both mean and fluctuation components representing the originally chaotic motion are stable, i.e., they are not sensitive to changes of initial conditions, and are fully reproducible.

Thus, chaos as a supersensitivity to initial conditions can be eliminated by describing the originally chaotic motion in an enlarged class of functions, for instance, by performing a Reynolds-type transformation and applying the stabilization principle. Nevertheless, the new deterministic representation will still contain an uncertainty coming from the "lack of knowledge" about initial conditions. However, this uncertainty has a subjective, rather than objective nature: as stressed by Ford (1988), randomness in chaotic motions is not an attribute of the dynamics itself, but rather a result of its mathematical treatment, i.e., chaos only makes predictions difficult, but not impossible. This view on chaos was recently corroborated by da Costa and Doria (1991), who, based upon Godel's incompleteness theorem, presented a rigorous proof of the algorithmic impossibility of deciding whether a given equation has chaotic domains or not in the class of elementary functions. Turning back to our original problem of unpredictability and irreversibility in Newtonian dynamics, one might ask now: are there some additional mathematical restrictions in Newtonian dynamics which do not have a solid enough physical ground? As shown in Zak (1991), there are such restrictions. One of them is the Lipschitz condition, which requires that for a dynamical system

$$\dot{x}_i = v_i(x_1, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{4}$$

all the derivatives

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty, \qquad i, j = 1, 2, \ldots, n \tag{5}$$

must be bounded.

This condition allows one to describe the Newtonian dynamics within the mathematical framework of the classical theory of differential equations, which guarantees its reversibility and predictability. That, in turn, leads to such effects as infinite time of approaching an attractor, infinite time for escape of a repeller if changes in initial conditions are infinitesimal [equations (1)–(3)], untractability of two trajectories which originally are very close but diverge exponentially, etc.

Hence, there are variety of phenomena whose explanations cannot be based directly upon the classical dynamics: they require in addition some words about a scale of observation, very close trajectories, etc.

Turning to governing equations of classical dynamics,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=\frac{\partial L}{\partial q_i}-\frac{\partial R}{\partial \dot{q}_i}, \qquad i=1, 2, \ldots, n \tag{6}$$

where L is the Lagrangian, q and $\dot{q}_i$ are the generalized coordinates and velocity, and R is the dissipation function, one should recall that the structure of $R(\dot{q}_i, \ldots, \dot{q}_n)$ is not prescribed by Newton's laws: some additional assumptions are to be made in order to define it. The natural assumption (which has been never challenged) is that these functions can be expanded in a Taylor series with respect to equilibrium states:

$$\dot{q}_i=0 \tag{7}$$

Obviously this requires the existence of the derivative

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_i}\right|<\infty \qquad \text{at} \quad \dot{q}_i \to 0 \tag{8}$$

The departure from that condition was proposed in Zak (1992), where the following dissipation function was introduced:

$$R=\frac{1}{k+1}\sum_i \alpha_i \left|\sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j\right|^{k+1} \tag{9}$$

in which

$$k=\frac{p}{p+2}<1, \qquad p \gg 1 \tag{10}$$

where p is a large, odd number.

By selecting large p, one can make k close to 1 so that equation (9) is almost identical to the classical one (when $k=1$) everywhere, excluding a small neighborhood of the equilibrium point $\dot{q}_j=0$, while at this point

$$\left|\frac{\partial^2 R}{\partial \dot{q}_i\, \partial \dot{q}_j}\right| \to \infty \qquad \text{at} \quad \dot{q}_j \to 0 \tag{11}$$

Hence, the condition (8) is violated; the friction force

$$F_i=-\frac{\partial R}{\partial \dot{q}_i} \tag{12}$$

grows sharply at the equilibrium point, and then it gradually approaches its classical value. This effect can be interpreted as a mathematical representation of a jump from static to kinetic friction.

It appears that this small difference between the friction forces at $k=1$ and $k<1$ leads to fundamental changes in Newtonian dynamics.

First, the time of approaching attractors as well as the time of escaping repellers becomes theoretically finite. Second, at repellers the solution becomes totally unpredictable within the deterministic mathematical framework, but it remains fully predictable in the probabilistic sense. In contrast to classical chaos, here the randomness is generated by the differential operator itself as a result of the failure of uniqueness conditions at the equilibrium points.

This paper is devoted to a discussion of the probabilistic properties of terminal dynamics, i.e., dynamics based upon terminal attractors and repellers where the Lipschitz conditions are violated. Some of these properties were already analyzed (Zak, 1988, 1989*b*,*c*, 1990*a*,*b*, 1991*a*,*b*) in connection with the modeling of information processing in neurodynamics. In this work our attention will be concentrated upon physical aspects of the problem.

2. FOUNDATIONS OF TERMINAL DYNAMICS

2.1. Terminal Attractors and Repellers

Terminal dynamics can be introduced as a set of nonlinear ordinary differential equations of the form

$$\dot{x}_i = v_i^k(x_i, x_2, \ldots, x_n), \qquad i=1,2,\ldots,n \tag{13}$$

in which

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty \tag{14}$$

and k is given by (10).

Since $k<1$, and therefore,

$$\left|\frac{\partial \dot{x}_i}{\partial x_j}\right| = k v^{k-1}(x_1, \ldots, x_n)\left|\frac{\partial v_i}{\partial x_i}\right| \to \infty \qquad \text{if} \quad \dot{x}_i \to 0 \tag{15}$$

the Lipschitz condition (5) is violated at all the equilibrium points

$$\dot{x}_i = 0 \tag{15'}$$

As in the classical case, the equilibrium points (15′) are attractors if the real parts of the eigenvalues of the matrix

$$m = \left\| \frac{\partial v_i}{\partial x_j} \right\| \tag{16}$$

are negative, that is,

$$\operatorname{Re} \lambda_i < 0 \tag{17}$$

and are repellers if some of the eigenvalues have positive real parts.

In order to emphasize the difference between classical and terminal equilibrium points, we will start with the simplest terminal dynamical system:

$$\dot{x} = -x^{1/3} \tag{18}$$

This equation has an equilibrium point at $x=0$ at which the Lipschitz condition (5) is violated:

$$\frac{d\dot{x}}{dx} = -\tfrac{1}{3} x^{-2/3} \to -\infty \qquad \text{at} \quad x \to 0 \tag{19}$$

Since here the condition (17) is satisfied

$$\operatorname{Re} \lambda \to -\infty < 0 \tag{20}$$

this point is an attractor of "infinite" stability.

The relaxation time for a solution with the initial condition $x = x_0 < 0$ to this attractor is finite:

$$t_0 = -\int_{x_0}^{x \to 0} \frac{dx}{x^{1/3}} = \tfrac{3}{2} x_0^{2/3} < \infty \tag{21}$$

Consequently, this attractor becomes terminal. It represents a singular solution which is intersected by all the attracted transients (Figures 1 and 2).

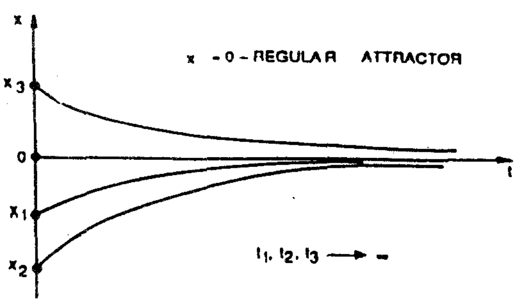

Fig. 1. Convergence to a regular attractor. $x=0$; t_1, t_2, $t_3 \to \infty$.

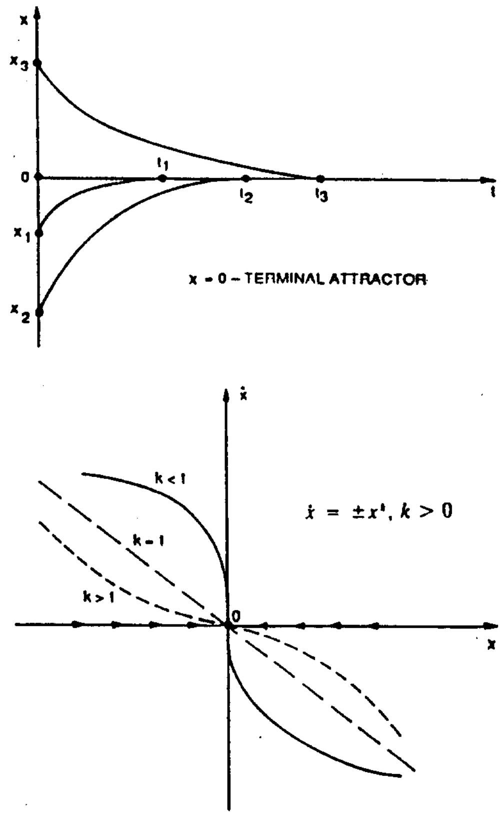

Fig. 2. Convergence to a terminal attractor. Top: $x=0$. Bottom: $\dot{x}=\pm x^k$, $k>0$.

For the equation

$$\dot{x}=x^{1/3} \tag{22}$$

the equilibrium point $x=0$ becomes a terminal repeller:

$$\frac{d\dot{x}}{dx}\rightarrow\tfrac{1}{3}x^{-2/3}\rightarrow\infty \quad \text{at} \quad x\rightarrow 0, \quad \text{i.e.,} \quad \operatorname{Re}\lambda\rightarrow\infty>0 \tag{23}$$

If the initial condition is infinitely close to this repeller, the transient solution will escape the repeller during a finite time period:

$$t_0=\int_{\varepsilon\rightarrow 0}^{x_0}\frac{dx}{x^{1/3}}=\tfrac{3}{2}x_0^{2/3}<\infty \quad \text{if} \quad x<\infty \tag{24}$$

while for a regular repeller, the time would be infinite [see equation (3)].

Instead of equations (18) and (22), one can consider a more general case:

$$\dot{x} = \pm x^k, \qquad k > 0 \tag{25}$$

for which the relaxation time (for the attractor) or the escaping time (for the repeller) is

$$t_0 \begin{cases} \to \infty & \text{if} \quad k \geq 1 \\ = x_0^{1-k}/(1-k) & \text{if} \quad k < 1 \end{cases} \tag{26}$$

As shown in the theory of differential equations, singular solutions in the equations

$$F(x, y, y') = 0 \tag{27}$$

are found by eliminating y' from the system:

$$F(x, y, y') = 0, \qquad \frac{\partial F}{\partial y'} = 0 \tag{28}$$

Hence, static terminal attractors [if they exist in equation (27)] must be among the solutions to the system (28).

2.2. Static Terminal Attractors with Terminal Trajectories

As shown in nonlinear dynamics, different types of regular attractors (or repellers) can be introduced based on the second-order dynamical system linearized with respect to the origin $x = 0$, $y = 0$:

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \tag{29}$$

or

$$\frac{dx}{dy} = \frac{ax + by}{cx + dy} \tag{30}$$

Depending upon the eigenvalues of the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{31}$$

the attractors (or repellers) $x = 0$, $y = 0$ can be a node, a star, a spiral point, or an improper node.

If instead of equation (30) one introduces the following system:

$$\dot{x} = (ax + by)^{1/3}$$
$$\dot{y} = (cx + dy)^{1/3} \tag{32}$$

Then the equilibrium point $x=0$, $y=0$ represents a terminal attractor (or repeller): the Lipschitz condition is violated at this point. Nevertheless, the differential equation of trajectories in configuration space x, y:

$$\frac{dx}{dy} = \left(\frac{ax+by}{cx+dy}\right)^{1/3} \tag{33}$$

satisfies the Lipschitz condition and it does not have any singular solutions. This means that both variables x and y are "simultaneously" approaching the terminal attractor, as in the case of a regular attractor [see equation (30)]. Moreover, a similar classification of terminal attractors of the type (32) can be performed, based upon the coefficients a, b, c, and d.

This section introduces a more "pathological" situation, when for the differential equations of trajectories in configuration space, the Lipschitz condition is also violated. As will be shown below, such a violation will lead to the loss of the uniqueness of the solutions in the configuration space: the trajectories will merge before approaching the terminal attractor.

Let us start with the following dynamical system:

$$\dot{x} = -(x - x^*)^{1/3} \tag{34}$$

$$\dot{y} = [y(x - x^*)]^{1/3} \tag{35}$$

It is easily verifiable that the Lipschitz condition here is violated at $x = x^*$, $y=0$. The differential equation of the trajectories in configuration space x, y can be written as

$$\frac{dy}{dx} = -y^{1/3} \tag{36}$$

For this equation, the Lipschitz condition is violated at $y=0$. This means that $y=0$ is a singular solution, and all the trajectories in configuration space x, y first flow to the x-axis, i.e., $y=0$, and then approach the terminal attractor $x=x^*$, $y=0$ together (Figure 3).

Indeed, it follows from equation (36) that

$$y = \begin{cases} (y_0^{2/3} - \frac{2}{3}x)^{3/2} & \text{for} \quad x \leq \frac{3}{2}y_0^{2/3} \\ 0 & \text{for} \quad x \geq \frac{3}{2}y_0^{2/3} \end{cases} \tag{37}$$

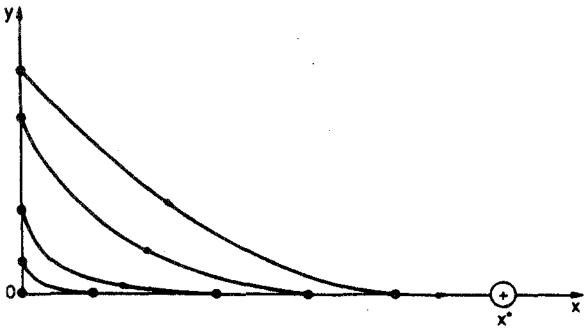

Fig. 3. Terminal trajectory $y=0$.

while $x=x(t)$ follows from equation (34):

$$x=\begin{cases} x^*+[(x_0-x^*)^{2/3}-\frac{2}{3}t]^{3/2} & \text{for} \quad t\leq\frac{3}{2}(x_0-x^*)^{2/3} \\ 0 & \text{for} \quad t\geq\frac{3}{2}(x_0-x^*)^{2/3} \end{cases} \tag{38}$$

Here x_0 and y_0 are the initial conditions.

The time of approaching the singular solution $y=0$ by the variable y follows from equation (37) if $x(t)$ is substituted from equation (38):

$$t_1=\tfrac{1}{3}(x_0-x^*)^{2/3} \tag{39}$$

The time t_1 of convergence of the solution to the terminal attractor follows from equation (38):

$$t_2=\tfrac{3}{2}[(x_0-x^*)^{2/3}-(\tfrac{3}{2}y_0^{2/3}-x^*)^{2/3}]\cdots \tag{40}$$

Obviously

$$t_2<t_1 \tag{41}$$

This means that the trajectory of the motion of the original dynamical system (34), (35) in the configuration space x, y first flows ino the trajectory $y=0$, and only then does it approach the terminal attractor $x=x^*$, $y=0$. Such a trajectory as $y=0$ we will call a terminal trajectory.

The situation described above can be generalized to the case where a terminal trajectory is a prescribed curve. Indeed, turning again to the system (34), (35), let us transfer to a new system of coordinates

$$x=\vartheta_1, \qquad y=f(\vartheta_1, \vartheta_2) \tag{42}$$

assuming that f is a differentiable function, and $\partial f/\partial\vartheta_2\neq 0$.

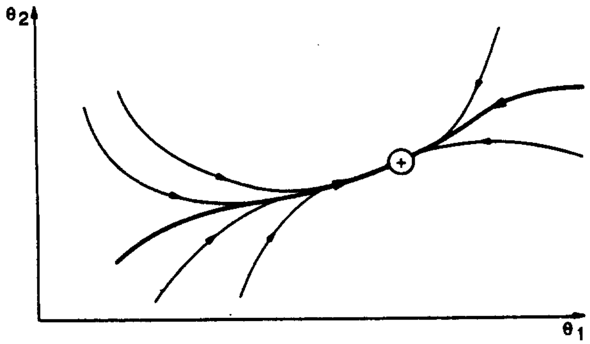

Fig. 4. Terminal trajectory $y=f(x)$.

Then equations (34) and (35) read

$$\dot{\vartheta}_1=-(\vartheta_1-\vartheta_1^*)^{1/3} \tag{43}$$

$$\dot{\vartheta}_2=\frac{1}{\partial f/\partial \vartheta_2}(\vartheta_1-\vartheta_1^*)^{1/3}\left(f^{1/3}+\frac{\partial f}{\partial \vartheta_1}\right)\cdots \tag{44}$$

The terminal trajectory $y=0$ is converted into a curve:

$$f(\vartheta_1, \vartheta_2)=0 \tag{45}$$

Hence, for a desired terminal trajectory (45), the corresponding dynamical system is (43), (44) (Figure 4).

2.3. Physical Interpretation of Terminal Attractors

As pointed out in the Introduction, the mathematical formalism of terminal dynamics follows from a more general structure of the dissipation function which allows the existence of smooth transitions from static to kinetic friction. It should be emphasized that the behavior of the solutions around the equilibrium points in terminal dynamics is more "realistic" than in the classical dynamics since the actual time of convergence to equilibrium points is finite. However, in order to make it finite, one has to violate the Lipschitz condition (5) since all the trajectories must intersect at the equilibrium point (Figure 2). In classical dynamics the Lipschitz condition (5) is not violated, and the infinite time of convergence is accounted for by some "small dissipative forces" which are always present. Actually terminal dynamics incorporates these forces via the parameter k [see equation (10)] which can be found from measurement of the convergence time [see equation (26)].

Terminal effects in fluid dynamics and their relevance to theory of turbulence were discussed in Zak (1992). It can be shown that the terminal attractor as a mathematical concept has other physical interpretations, and one of them is the energy-cumulation effect, although in this case one deals with the finite time of convergence of a propagating wave rather than the motion of an individual particle.

As an example, consider the propagation of an isolated pulse in an elastic continuum along the x axis. In general, the speed of propagation $\dot{x} = \lambda$ depends on x. Suppose there exists such a point x^* where $\lambda(x^*) = 0$. Then the time t^* during which the leading front of the propagating pulse will approach this point is expressed via the following integral:

$$t^* = \int_{x_0}^{x \to x^*} \frac{dx}{\lambda(x)} \tag{46}$$

If λ can be presented in the form

$$\lambda = (x^* - x)^k, \qquad 0 < k < 1 \tag{47}$$

then this integral converges and, therefore, the time t^* is finite. It is easily verifiable that in this case the differential equation

$$\dot{x} = (x^* - x)^k \tag{48}$$

describing the dynamics of the pulse propagation has a terminal attractor at $x = x^*$. But if the leading and the trailing fronts of the propagating pulse approach the same point x^* during a finite time, then eventually the width of the pulse will shrink to zero, and all the energy transported by the pulse will be distributed over a vanishingly small length. Hence, the existence of a terminal attractor in such models leads to an unbounded concentration of energy in the neighborhood of the attractor.

Based upon this model, Zak (1970, 1982*c*, 1983) explained and described the formation of a supersonic snap at the free end of a filament suspended in a gravity and a centrifugal force field, as well as the cumulation of the shear strain energy at the soil surface in response to an underground explosion. In these models, the free end of the filament and the free surface of the soil serve as terminal attractors.

Some terminal effects in fluid dynamics were introduced and discussed in Zak (1992).

2.4. Periodic Terminal Limit Sets

So far, this paper has concentrated on static terminal attractors. I will now demonstrate the existence of periodic terminal attractors. For that purpose, let us consider a dynamical system separable in polar coordinates r, θ:

$$\dot{r}=r(R-r)^{1/3}, \qquad r\leq R \tag{49}$$

$$\dot{\theta}=\omega \tag{50}$$

Here, $d\dot{r}/dr \rightarrow -\infty$ at $r \rightarrow R$ [compare with equation (19)] and therefore the solution $r=R$, $\theta=\omega t+\theta_0$ is a terminal limit cycle. Its basin is defined by the condition $r>0$. For the solution with the initial condition $r_0<R$ the relaxation time is finite:

$$t_0=\int_{r_0}^{R}\frac{dr}{r(R-r)^{1/3}}<\int_{r_0}^{R}\frac{dr}{r_0(R-r)^{1/3}}=\frac{2}{3r_0}(R-r_0)^{2/3}<\infty \tag{51}$$

It is easily verifiable that a periodic terminal repeller can be obtained by changing the sign in the right-hand side of equation (49).

The terminal analog of chaotic attractor was introduced and discussed in Zak (1991*a*).

2.5. Unpredictability in Terminal Dynamics

The concept of unpredictability in classical dynamics was introduced in connection with the discovery of chaotic motions in nonlinear systems. Such motions are caused by the Lyapunov instability, which is characterized by a violation of the continuous dependence of solutions on the initial conditions during an unbounded time interval ($t \rightarrow \infty$). That is why the unpredictability in these systems develops gradually. Indeed, if two initially close trajectories diverge exponentially,

$$\varepsilon=\varepsilon_0 \exp \lambda t, \qquad 0<\lambda<\infty \tag{52}$$

then for an infinitesimal initial distance $\varepsilon_0 \rightarrow 0$, the current distance ε becomes finite only at $t \rightarrow \infty$. For this reason, the Lyapunov exponents (the mean exponential rate of divergence) are defined in an unbounded time interval:

$$\sigma=\lim\left(\frac{1}{t}\right)\ln\frac{\varepsilon}{\varepsilon_0}, \qquad t\rightarrow\infty \tag{53}$$

In distributed dynamical systems, described by partial differential equations, there exists a stronger instability discovered by Hadamard. In the course of this instability, a continuous dependence of a solution on the initial conditions is violated during an arbitrarily small time period. Such a blowup instability is caused by a failure of hyperbolicity and transition to ellipticity (Zak, 1982*a*–*c*).

This section will show that a similar type of blowup instability leading to "discrete pulses" of unpredictability can occur in dynamical systems which contain terminal repellers (Zak, 1989*b*).

Let us analyze the transient escape from the terminal repeller in the equation

$$\dot{x} = x^{1/3}, \qquad x_0 = x(0) \tag{54}$$

assuming that $|x_0| \to 0$. The solution to equation (4) reduces to

$$x = \pm t^{3/2}, \qquad x \neq 0 \tag{55}$$

Hence, two different solutions are possible for "almost the same" initial conditions [compare to equation (3)]. The most essential property of this result is that the divergence of the solutions (55) is characterized by an unbounded parameter which can be associated with a terminal version of the Lyapunov exponent:

$$\sigma = \lim_{t \to t_0} \left(\frac{1}{t} \ln \frac{2t^{3/2}}{2|x_0|} \right) = \infty, \qquad |x_0| \to 0 \tag{56}$$

where t_0 is an arbitrarily small (but finite) positive quantity. In contrast to equation (36), here the terminal Lyapunov exponent can be defined in an arbitrarily small time interval, since during this interval the initial infinitesimal distance between the solutions becomes finite. Thus, a terminal repeller represents a vanishingly short but infinitely powerful "pulse of unpredictability" which is "pumped" into the dynamical system.

In order to illustrate the unpredictability in such a non-Lipschitzian dynamics, we turn to the following equation:

$$\dot{x} - yx^{1/3} = 0 \tag{57}$$

while

$$y = \cos \omega t \tag{58}$$

Assuming that $x \to 0$ at $t \to 0$, one obtains regular solutions

$$x = \pm \left(\frac{2}{3\omega} \sin \omega t \right)^{3/2}, \qquad x \neq 0 \tag{59}$$

and a singular solution (an equilibrium point)

$$x = 0 \tag{60}$$

During the first time period

$$0 < t < \frac{\pi}{2\omega} \tag{61}$$

the equilibrium point (60) is a terminal repeller (since $y > 0$). Therefore, within this period, the solutions (59) have the same property as the solutions (55): their divergence is characterized by an unbounded Lyapunov exponent.

During the next time period

$$\frac{\pi}{2\omega} < t < \frac{3\pi}{2\omega}$$

the equilibrium point (60) becomes a terminal attractor (since $y < 0$), and the system which approaches this attractor at $t = \pi\omega$ remains motionless until $t > 3\pi/2\omega$. After that, the terminal attractor converts into the terminal repeller, and the system escapes again, etc.

It is important to notice that each time the system escapes the terminal repeller, the solution splits into two symmetric branches, so that the total trajectory can be combined from 2^n pieces, where n is the number of cycles, i.e., it is the integer part of the quantity $(t/2\pi\omega)$. As one can see, here the nature of the unpredictability is significantly different from the unpredictability in chaotic systems.

One can notice that the motion (59) resembles chaotic oscillations known from classical dynamics: it combines random characteristics with the attraction to a center. However, in contrast to classical chaos, the motion (59) is driven by the failure of the uniqueness of the solution at the equilibrium point, and it has a well-organized probabilistic structure. Since the time of approaching the equilibrium point $x = 0$ by the solution (59) is finite, this type of chaos can be called terminal (Zak, 1991*a*, 1992).

2.6. Irreversibility of Terminal Dynamics

Classical dynamics describe processes in which time t plays the role of a parameter: it remains fully reversible in the sense that the time-backward motion can be obtained from the governing equation by time inversion, $t \to -t$. This means that classical dynamics cannot explain the emergence of new dynamical patterns in nature.

However, there exists another class of phenomena where past and future play different roles, and time is not invertible: by definition (the second law of thermodynamics) irreversibility is introduced in thermodynamics by postulating the increase of entropy.

As stressed by Prigogine (1980), irreversible processes play a fundamental constructive role in the physical world; they are at the basis of important coherent processes that appear with particular clarity on the biological level.

In this connection let us compare the dynamical behavior of solutions in small neighborhoods of classical and terminal repellers:

$$\dot{x}=x \tag{62}$$

and

$$\dot{x}=x^{1/3} \tag{63}$$

The solution to equation (62),

$$x_{+}=x_0\, e^{t} \tag{64}$$

describing an escape from a classical repeller is reversible since

$$u_{-}=x_0\, e^{-t} \tag{65}$$

is a possible motion describing the convergence to a classical attractor $x=0$.

The solution to equation (63)

$$x_{+}=\sqrt{(\tfrac{2}{3}t)^3} \tag{66}$$

is irreversible since the time-backward motion

$$x_{-}=\sqrt{-(\tfrac{2}{3}t)^3} \tag{67}$$

does not exist (x has imaginary value).

This mathematical formalism expresses the deeper roots of irreversibility of terminal dynamics, which can be understood if one turns to the solution of dynamics (57), (58). This solution consists of regular (59) and singular (60) parts. When the regular solution approaches the equilibrium point $x=0$ (in finite time), it switches to the singular solution $x\equiv 0$, and this switch is irreversible.

3. PROBABILISTIC STRUCTURE OF TERMINAL DYNAMICS

As shown in Zak (1992), the terminal version of Newtonian dynamics is different from its classical version only within vanishingly small neighborhoods of equilibrium states, and therefore it contains classical mechanics as a special case. This means that terminal dynamics is not necessarily always unpredictable and irreversible: in some domains it is identical with the classical dynamics. However, in this section our attention will be concentrated on specific effects of terminal dynamics, and in particular, on its probabilistic structure.

One should emphasize again the fundamental difference between probabilistic properties of terminal dynamics and those of stochastic or chaotic differential equations. Indeed, the randomness of stochastic differential equations is caused by random initial conditions, random forces, or random coefficients; in chaotic equations small (but finite!) random changes of initial conditions are amplified by the mechanism of instability. But in both cases the differential operator itself remains deterministic. In contrast, in terminal dynamics randomness results from the violation of the uniqueness of the solution at equilibrium points, and therefore the differential operator itself generates random solutions.

3.1. Terminal Version of Liouville–Gibbs Theorem

The Liouville–Gibbs theorem in classical dynamics expresses the relationship between the governing differential equations and equations for probability distribution functions. For the dynamical system

$$\dot{x}_i = v_i(x_1, x_2, \ldots, x_n), \qquad i = 1, 2, \ldots, n \tag{68}$$

it has two equivalent forms:

$$\frac{\partial f}{\partial t} + \operatorname{div}(f v_i) = 0 \tag{69}$$

or

$$f = f_0 \exp\left(-\int_0^t \operatorname{div} v_i \, dt\right) \tag{70}$$

Here x_i are considered as random variables, while randomness is introduced only through initial conditions x_i^0 possessing a given joint distribution with the joint density f_0, and f is the current joint distribution. Since the operator v_i is deterministic, the system (68) can be solved in a deterministic way, and to make the solution vector x a random vector, it suffices to treat the initial conditions as random variables.

It should be recalled that equations (69) and (70) were derived for the case when the Lipschitz conditions are satisfied, i.e.,

$$\left|\frac{\partial \dot{x}_i}{\partial x_j}\right| < \infty, \qquad i, j = 1, 2, \ldots, n \tag{71}$$

which means that

$$\left|\frac{\partial v_i}{\partial x_j}\right| < \infty, \qquad i, j = 1, 2, \ldots, n \tag{72}$$

For the terminal dynamics given by equations (13) these conditions do not hold [see equation (15)]. In addition, one can verify that

$$|\text{div } v_i| = k \left| \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} v_i^{k-1} \right| \to \infty \qquad \text{if} \quad \dot{x}_j \to 0 \tag{73}$$

i.e., $|\text{div } v_i|$ is unbounded at equilibrium points.

This means that equations (69) and (70) are valid everywhere, excluding the equilibrium points $\dot{x}_j = 0$.

In order to define the distribution f at $\dot{x}_j = 0$, we first evaluate the functions

$$x_i = x_i(t) \qquad \text{at} \quad x_i \to \overset{*}{x}_i = 0 \tag{74}$$

where $\overset{*}{x}_i$ are the coordinates of an equilibrium point.

For simplicity (but without loss of generality) we assume that

$$\overset{*}{x}_i = 0 \tag{75}$$

Then

$$\dot{x}_i \sim (\alpha_i x_i)^k \qquad \text{at} \quad x_i \to 0, \quad x_{j \neq i} = 0 \tag{76}$$

where

$$\alpha_i = \frac{\partial v_i}{\partial x_i} \qquad \text{at} \quad x_i = 0, \quad x_2 = 0, \ldots, \quad x_n = 0 \tag{77}$$

Let us assume first that

$$\alpha_i > 0 \tag{78}$$

i.e., the equilibrium point (75) is a terminal repeller. Then in a small neighborhood of this point

$$x_i \sim t^{1/(1-k)} \qquad \text{at} \quad x_i \to 0 \tag{79}$$

and therefore

$$\text{div } v_i \sim k \sum_i (\alpha_i x_i)^{k-1} \sim \frac{1}{t} \qquad \text{at} \quad x_i \to 0 \tag{80}$$

Hence

$$-\int_0^t \text{div } v_i \, dt = \int_t^{\varepsilon \to 0} \text{div } v_i \, dt \sim \ln \frac{\varepsilon \to 0}{t} \qquad \text{at} \quad x_i \to 0 \tag{81}$$

i.e.,

$$\exp\left(-\int_0^t \operatorname{div} v_i \, dt\right) \sim \frac{\varepsilon \to 0}{t} \to 0 \qquad \text{at} \quad x_i \to 0 \tag{82}$$

Therefore, it follows from (70) that

$$f \to 0 \qquad \text{at} \quad x_i \to 0 \quad \text{if} \quad f_0 < \infty \tag{83}$$

This means that those trajectories which originated outside of the terminal repeller will never approach it; it follows from (83) that the terminal repeller generates probability even if the initial conditions are "almost" deterministic. In other words, it represents a "vacuum" of the probability density.

For a terminal attractor, i.e., when

$$\alpha_i < 0 \tag{84}$$

after following transformations similar to those performed in (79)–(82), one obtains

$$f \to \infty \qquad \text{at} \quad x_i \to 0 \tag{85}$$

Hence, those trajectories which originated outside of the terminal attractor will definitely approach it, i.e., the terminal attractor represents a center of concentration of all the probability "mass."

3.2. Terminal Model of Random Walk

A random walk is a stochastic process where changes occur only at fixed times. In this section we introduce the terminal dynamics which describes this process.

Let us start with the following dynamical system:

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t, \qquad \gamma = \text{const}, \quad \omega = \text{const}, \quad \alpha = \text{const} \tag{86}$$

It can be verified that at the equilibrium points

$$x_m = \frac{\pi m \alpha}{\sqrt{\omega}}, \qquad m = \cdots, -2, -1, 0, 1, 2, \ldots \tag{87}$$

the Lipschitz condition is violated:

$$\partial \dot{x}/\partial x \to \infty \qquad \text{at} \quad x \to x_m \tag{88}$$

If $x=0$ at $t=0$, then during the first period

$$0<t<\pi/\omega \tag{89}$$

the point $x_0=0$ is a terminal repeller since $\sin \omega t>0$ and the solution at this point splits into two (positive and negative) branches whose divergence is characterized by an unbounded terminal Lyapunov exponent [see equation (56)]. Consequently, with an equal probability, x can move into the positive or the negative direction. For the sake of concreteness, we will assume that it moves in the positive direction. Then the solution will approach the second equilibrium point $x_1=\pi\alpha/\sqrt{\omega}$ at

$$t^*=\frac{1}{\omega}\arccos\left[1-\frac{B(\frac{1}{3},\frac{1}{3})}{2^{1/3}}\frac{\alpha\sqrt{\omega}}{\gamma}\right] \tag{90}$$

in which B is the beta function.

It can be verified that the point x_1 will be a terminal attractor at $t=t_1$ if

$$t_1\leq\frac{\pi}{\omega},\qquad \text{i.e.,}\qquad \text{if}\quad \frac{\gamma}{\alpha}\geq\frac{B(\frac{1}{3},\frac{1}{3})}{2^{4/3}}\sqrt{\omega} \tag{91}$$

Therefore, x will remain at the point x_1 until it becomes a terminal repeller, i.e., until $t>t_1$. Then the solution splits again: one of two possible branches approaches the next equilibrium point $x_2=2\pi\alpha/\sqrt{\omega}$, while the other returns to the point $x_0=0$, etc. The periods of transition from one equilibrium point to another are all the same and are given by equation (90) (Figure 5).

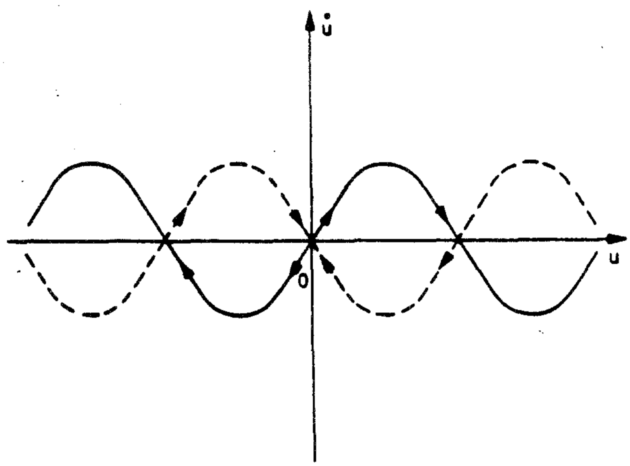

Fig. 5. Oscillations about the attractor $\dot{u}=0$.

It is important to notice that these periods t^* are bounded only because of the failure of the Lipschitz condition at the equilibrium points. Otherwise they would be unbounded, since the time of approaching a regular attractor (as well as the time of escaping a regular repeller) is infinite.

Thus, the evolution of x prescribed by equation (86) is totally unpredictable: it has 2^m different scenarios, where $m = E(t/t^*)$ (Figure 6), while any prescribed value of x from equation (87) will appear eventually. This evolution is identical to a random walk, and the probability $f(x, t)$ is governed by the following difference equation:

$$f\left(x, t+\frac{\pi}{\omega}\right) = \tfrac{1}{2} f\left(x - \frac{\pi\alpha}{\sqrt{\omega}}, t\right) + \tfrac{1}{2} f\left(x + \frac{\pi\alpha}{\sqrt{\omega}}, t\right) \tag{92}$$

For better physical interpretation we will assume that

$$\frac{\pi\alpha}{\sqrt{\omega}} \ll L, \; t^* \ll T, \qquad \text{i.e.,} \quad \omega \to \infty \tag{93}$$

in which L and T are the total length and the total time period of the random walk, respectively. Setting

$$\frac{\pi\alpha}{\sqrt{\omega}} \to 0, \qquad t^* \to 0 \tag{94}$$

one arrives at the Fokker–Planck equation:

$$\frac{\partial f(x, t)}{\partial t} = \tfrac{1}{2} D^2 \frac{\partial^2 f(x, t)}{\partial x^2}, \qquad D^2 = \pi\alpha^2 \tag{95}$$

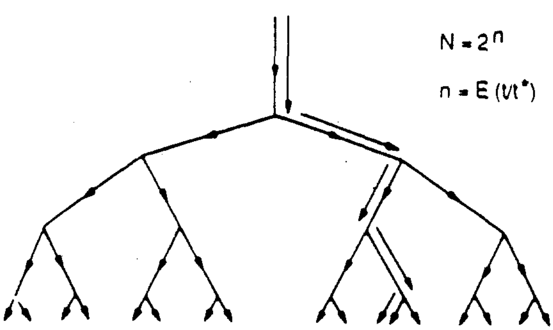

Fig. 6. Unpredictable system.

Its unrestricted solution for the initial condition that the random walk starts from the origin $x=0$ at $t=0$,

$$f(x, t)=\frac{1}{\sqrt{(2\pi D^2 t)}}\exp\left(-\frac{x^2}{2D^2 t}\right) \tag{96}$$

qualitatively describes the evolution of the probability distribution for the dynamical equation (86). It is worth noticing that for the exact solution one should turn to the difference equation (92), since actually $\omega<\infty$.

3.3. Probabilistic Attractors in Terminal Dynamics

In this section we describe a new dynamical effect—an attraction with a certain probability to a point. For this purpose we slightly modify equation (86):

$$\dot{x}=\gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha} y(x)\right]\sin \omega t \tag{97}$$

assuming that

$$y'=\frac{dy}{dx}>\beta>0, \qquad \beta=\text{const} \tag{98}$$

and

$$k=\frac{1}{2n+1}, \qquad n\to\infty \tag{99}$$

where n is an integer.

This replacement does not change the qualitative behavior of the dynamical system (97): it changes only its quantitative behavior between the critical points in such a way that one has explicit control over the period of transition from one critical point to another. Indeed, since

$$\lim \sin^{1/2n+1}_{n\to\infty} X=\text{sgn}\sin X \tag{100}$$

one obtains the solution for x which is valid between critical points $x^{(m)}$ and $x^{(m+1)}$:

$$x=\frac{\gamma}{\omega}(1-\cos \omega t) \tag{101}$$

Obviously the distances between the equilibrium points will depend upon the step m:

$$h_m = x_m - x_{m-1} = y^{-1}\left(\frac{\pi \alpha m}{\sqrt{\omega}}\right) - y^{-1}\left(\frac{\pi \alpha (m-1)}{\sqrt{\omega}}\right) \tag{102}$$

where $y^{-1}(x)$ is the inverse of $y(x)$.

The period of transition from the $(m-1)$th to the mth critical point follows from (101) and (102):

$$t^* = \frac{1}{\omega} \arccos\left(1 - \frac{h_m}{\gamma}\right) \leq \frac{\pi}{\omega} \tag{103}$$

i.e.,

$$\gamma \geq \omega h_m, \qquad m = 1, 2, \ldots \tag{104}$$

since it should not exceed the period between the conversions of terminal attractors into terminal repellers and vice versa.

Now instead of equation (92) one obtains

$$f\left(x, t + \frac{\pi}{\omega}\right) = 0.5 f(x - h_m, t) + 0.5 f(x + h_m, t) \tag{105}$$

in which h_m is given by equation (102).

Introducing a new variable $y(x)$ and substituting it into equation (97),

$$\dot{y} = \frac{\gamma}{y'} \sin^k \frac{\sqrt{\omega}}{\alpha} y \sin \omega t \tag{106}$$

one reduces equation (105) to the form of (92):

$$f\left(y, t + \frac{\pi}{\omega}\right) = 0.5 f\left(y - \frac{\pi \alpha}{\sqrt{\omega}}, t\right) + 0.5 f\left(y + \frac{\pi \alpha}{\sqrt{\omega}}, t\right) \tag{107}$$

For large (but bounded) ω, the continuous approximation of equation (95)

$$\frac{\partial f(y, t)}{\partial t} = \tfrac{1}{2} D^2 \frac{\partial^2 f(y, t)}{\partial y^2}, \qquad D^2 = \pi \alpha^2 \tag{108}$$

describes qualitatively the random walk (97)

$$f(y, t) = \frac{1}{\sqrt{2\pi D^2 t}} \exp\left(-\frac{y^2}{d^2 t}\right) \tag{109}$$

or, after returning to the old variable x,

$$f(x, t) = \frac{|y'(x)|}{\sqrt{2\pi D^2 t}} \exp\left[-\frac{y^2(x)}{2D^2 t}\right] \tag{110}$$

Let us assume that

$$y = \sqrt[2n+1]{x-1}, \qquad n \to \infty \tag{111}$$

Then

$$|y'(x)| \to \infty \qquad \text{if} \quad x \to 1 \tag{112}$$

and therefore

$$f \to \delta(x-1) \qquad \text{at} \quad t \to \infty \tag{113}$$

Hence, the solution to the dynamical equation (97) is a random function which is attracted to the point $x = 1$ with the probability

$$p \to 1 \qquad \text{at} \quad t \to \infty \tag{114}$$

irrespective of the initial probability distribution.

That is why such a point can be called a probabilistic attractor in terminal dynamics.

3.4. Guided Systems and Stochastic Attractors

Turning to equation (86), let us assume that this dynamical system is driven by a vanishingly small input $\varepsilon(t)$:

$$\dot{x} = \gamma \sin^{1/3} \frac{\sqrt{\omega}}{\alpha} x \sin \omega t + \varepsilon(t), \qquad |\varepsilon(t)| \ll \gamma \tag{115}$$

This input can be ignored when $\dot{x} \neq 0$, or when $\dot{x} = 0$, but the system is stable, i.e., $x = \pi\alpha/\sqrt{\omega}, 3\pi\alpha/\sqrt{\omega}, \ldots$.

However, it becomes significant during the instances of instability when $\dot{x} = 0$ at $x = 0$, $2\pi/\sqrt{\omega}$, etc. Since actually a vanishingly small noise is always present, one can interpret the unpredictability discussed above as a consequence of small random inputs to which the dynamical system (115) is extremely sensitive.

However, the function $\varepsilon(t) \ll \delta$ is not necessarily random: it can be associated with a device which controls the behavior of the dynamical system (86) through a string of signs. Indeed, the only important part in this point is the sign of $\varepsilon(t)$ at the critical points. Consider, for example, equation (115), and suppose that

$$\operatorname{sgn} \varepsilon(t_m) = +, +, -, +, -, -, \text{etc.} \quad \text{at} \quad t_m = \frac{\pi m}{\omega}, \quad m = 1, 2, \ldots \tag{116}$$

The values of $\varepsilon(t)$ in between the critical points are not important since, by our assumption, they are small in comparison to values of the derivative $\dot{x}$, and therefore can be ignored. Hence, the only part of the input $\varepsilon(t)$ which is significant in determining the solution to equation (115) is the sign of the string (116): specification of this string fully determines the dynamics of (115). Figure 7 demonstrates three different scenarios of motion for different strings. Such guided terminal dynamical systems were introduced and analyzed in Zak (1989*d*, 1990*a*,*b*, 1991*a*). In this paper we discuss more complex dynamical systems when the string (116) is undetermined in some critical points. We assume that

$$\varepsilon(t) = \varepsilon_0 a x, \qquad \varepsilon_0 \to 0 \tag{117}$$

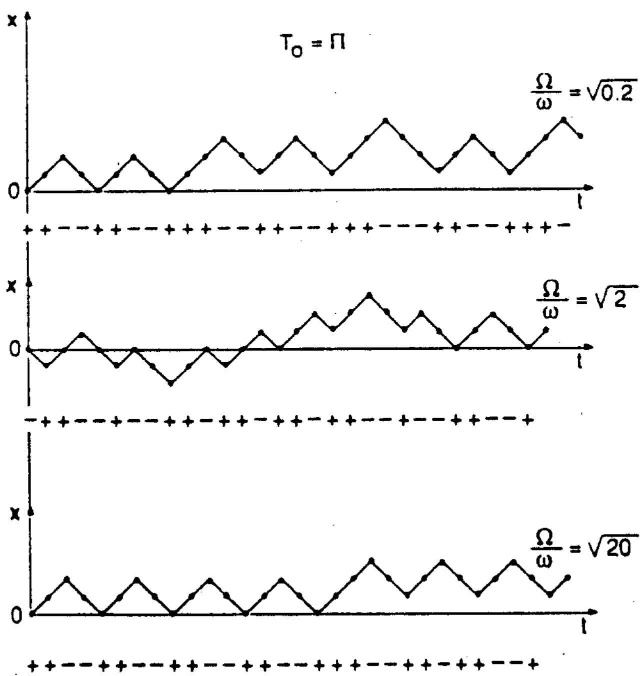

Fig. 7. Temporal patterns and their codes.

where

$$a\begin{cases}=0 & \text{if} \quad 1<x<-1\\ <0 & \text{if} \quad x=1\\ >0 & \text{if} \quad x=-1\end{cases} \tag{118}$$

The conditions (118) can be implemented via the additional terminal dynamical system

$$\dot{a}=a^{1/3}(x-1)(x+1)-\varepsilon_0 a, \qquad \varepsilon_0 \to 0 \tag{119}$$

Indeed, equation (119) has a terminal equilibrium point

$$a=0 \tag{120}$$

which is a terminal attractor if

$$-1<x<1 \tag{121}$$

and it is a terminal repeller otherwise. One can also verify that the solution escapes the terminal repeller such that

$$\begin{matrix} a<0 & \text{if} \quad x=1\\ a>0 & \text{if} \quad x=-1\end{matrix} \tag{122}$$

Hence, the dynamical system (119) fully implements the conditions (118).

Now let us return to equation (115) supplemented by equations (117) and (119). Within the domain (121) the solution describes a random walk governed by equation (92) or by its continuous approximation (95), and it is fully reflected from the boundaries $x=x_1$ and $x=x_2$. Indeed, it follows from equation (122) that

$$\operatorname{sgn} \varepsilon(t)=\begin{cases}- & \text{at} \quad x=1\\ + & \text{at} \quad x=-1\end{cases} \tag{123}$$

Hence, one arrives at a restricted random walk with boundary conditions

$$\left.\frac{\partial f}{\partial x}\right|_{x=1}=\left.\frac{\partial f}{\partial x}\right|_{x=-1}=0 \tag{124}$$

The solution to equation (95) subject to the boundary conditions (124) at $t \to \infty$ is

$$f(x)=0.5, \qquad -1<x<1 \tag{125}$$

i.e., with the same probability the solution will visit all the critical points within the domain (121).

Modifying equation (115) in the same way as in the previous section by introducing the new variable $y = y(x)$

$$\dot{x} = \gamma \sin^k\left[\frac{\sqrt{\omega}}{\alpha}\, y(x)\right] \sin \omega t + \varepsilon(t), \qquad |\varepsilon(t)| \ll \gamma \tag{126}$$

with the supplemental equations

$$\varepsilon(t) = \varepsilon_0 a y, \qquad \dot{a} = a^{1/3}(y-1)(y+1) - \varepsilon_0 a \tag{127}$$

one arrives at the following probability distribution instead of (125):

$$f(x) = 0.5|y'(x)|, \qquad y(-1) < x < y(1) \tag{128}$$

The solution (128) represents a stationary stochastic process which is an attractor of the dynamical system (115).

3.5. Multidimensional Systems

The results presented in the previous sections can be generalized to multidimensional dynamics.

We start with the following terminal dynamical system:

$$\dot{x}_i = \gamma_i \sin^k\left(\frac{\sqrt{\omega}}{\alpha_i} \sum_j T_{ij} x_j\right) \sin \omega t, \qquad T_{ij} = T_{ji}, \quad i, j = 1, 2 \tag{129}$$

assuming that $|T_{ij}|$ is a positive-definite matrix, i.e.,

$$T_{11} > 0, \qquad \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} > 0 \tag{130}$$

Here k is defined by equation (99). The property (130) provides stability (if $\sin \omega t < 0$) or instability (if $\sin \omega t > 0$) of the system (129) at the terminal equilibrium points:

$$\overset{*}{x}_1 = \frac{\pi \alpha_1}{\Delta\sqrt{\omega}} \begin{vmatrix} m_1 & T_{12} \\ m_2 & T_{12} \end{vmatrix}, \qquad \overset{*}{x}_2 = \frac{\pi \alpha_2}{\Delta\sqrt{\omega}} \begin{vmatrix} T_{11} & m_1 \\ T_{12} & m_2 \end{vmatrix}, \qquad \Delta = \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} \tag{131}$$

Here m_i is the number of steps made by the variable x_i. The coordination between the period of transition t^* from one critical point to another and the period between the conversions of terminal attractors into terminal repellers are provided by the condition

$$\gamma_i \geq \omega x_i, \qquad i = 1, 2 \tag{132}$$

The system (129) describes a two-dimensional random walk, and the joint density function $f(x_1, x_2, t)$ is governed by the following difference equation:

$$4f\left(x_1, x_2, t+\frac{\pi}{\omega}\right)=f(x+h_{11}, x_2+h_{22}, t)+f(x_1-h_{12}, x_2-h_{21}) \quad (133)$$
$$+f(x_1-h_{12}, x_2+h_{21})+f(x_1-h_{11}, x_2-h_{22})$$

or by its continuous approximation

$$\frac{\partial f}{\partial t}=\frac{1}{2}\left(D_{11}\frac{\partial^2 f}{\partial x_1^2}+D_{12}\frac{\partial^2 f}{\partial x_1\,\partial x_2}+D_{22}\frac{\partial^2 f}{\partial x_2^2}\right) \quad (134)$$

where

$$D_{11}=\frac{\pi}{\Delta^2}(\alpha_1^2T_{22}^2+\alpha_2^2T_{12}^2)$$

$$D_{12}=\frac{2\pi T_{12}}{\Delta^2}(\alpha_1^2T_{22}+\alpha_2^2T_{11})$$

$$D_{22}=\frac{\pi}{\Delta^2}(\alpha_2^2T_{11}^2+\alpha_1^2T_{12}^2)$$

$$h_{11}=\frac{\pi}{\Delta\sqrt{\omega}}(\alpha_1T_{22}-\alpha_2T_{12}), \qquad h_{21}=\frac{\pi}{\Delta\sqrt{\omega}}(\alpha_2T_{11}+\alpha_1T_{12})$$

$$h_{12}=\frac{\pi}{\Delta\sqrt{\omega}}(\alpha_1T_{22}+\alpha_2T_{12}), \qquad h_{22}=\frac{\pi}{\Delta\sqrt{\omega}}(\alpha_2T_{11}-\alpha_1T_{12})$$

As the one-dimensional case, the effect of probabilistic attraction can be incorporated into terminal dynamics by the introduction of new variables in equation (129):

$$\dot{x}_i=\gamma\sin^k\left[\frac{\sqrt{\omega}}{\alpha_i}y_i(\xi_i)\right]\sin\omega t, \qquad \xi=\sum_j T_{ij}x_j \quad (135)$$

assuming that y_i satisfies the condition (98).

Indeed, the point with the largest components $|y'(\xi)|$ will attract (in the probabilistic sense) the solution to equations (129) in the same way as described by equations (110)–(114).

The guided version of equation (135) can be represented as

$$\dot{x}_i = \gamma_i \sin^k \left[\frac{\sqrt{\omega}}{\alpha_i} y_i(\xi_i) \right] \sin \omega t + \varepsilon_i(t), \quad \xi_i = \sum_j T_{ij} x_j \tag{136}$$

where

$$\varepsilon_i(t) = \varepsilon_0 \sum_j a_{ij} x_j, \qquad \varepsilon_0 \to 0 \tag{137}$$

As in the one-dimensional case (117), the coefficients a_{ij} can be given by inequalities of the type (118) by means of additional terminal dynamics of the type (119). This will lead to a multidimensional random walk restricted by reflecting boundaries, while the limiting form of the solution at $t \to \infty$ represents a stationary stochastic attractor of the dynamical system (135).

Thus, there are two kinds of coupling between the variables x_i in the terminal dynamics (136): the coefficients T_{ij} carry out a probabilistic coupling via the joint density distribution, while the coefficients a_{ij} perform a deterministic, but "qualitative" rather than quantitative coupling [since only the sign of $\varepsilon_i(t)$ at the terminal equilibrium points is important].

4. CONCLUSION

One of the major flaws in Newtonian dynamics is its determinism and reversibility, which makes it impossible to explain the emergence of new dynamical patterns in nature in the way in which nonequilibrium thermodynamics does. However, in our view, both of these characteristics are attributes of the mathematical models of Newtonian dynamics rather than Newton's laws themselves. Indeed, these models require some additional restrictions for the sake of mathematical convenience, and some of them are not always consistent with the physical nature of the motion. One such restriction is the Lipschitz condition (which is responsible for the determinism and reversibility of Newtonian dynamics). Indeed, all real physical systems approach their equilibria in a finite time. That can occur only due to static friction, which does not vanish with the velocity. In mathematical language, this means that the Lipschitz condition at these points is violated.

In this paper a new mathematical model for Newtonian dynamics—the terminal dynamics—is introduced and analyzed. This model reshapes the dissipation function in such a way that the time of approaching equilibrium points becomes theoretically finite due to the violation of the Lipschitz condition. As a "side effect" of this property, terminal dynamics becomes irreversible and probabilistic.

The paper discusses the foundations and probabilistic structure of terminal dynamics, and, in particular, a new phenomenon—a probabilistic dynamical attractor.

One of the most significant properties of terminal dynamics is that it can impose upon its variables several types of nonrigid constraints such as probabilistic coupling via the joint density, or "qualitative" coupling via the sign of certain combinations of variables. Both of these constraints are very pronounced in biological and social systems, which are characterized by the possibility of emergence of new dynamical patterns.

Thus, terminal dynamics can become a powerful mathematical tool for modeling irreversible and nondeterministic processes in nature. At the same time it allows us to reevaluate our view on such random phenomena as turbulence and chaos.

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